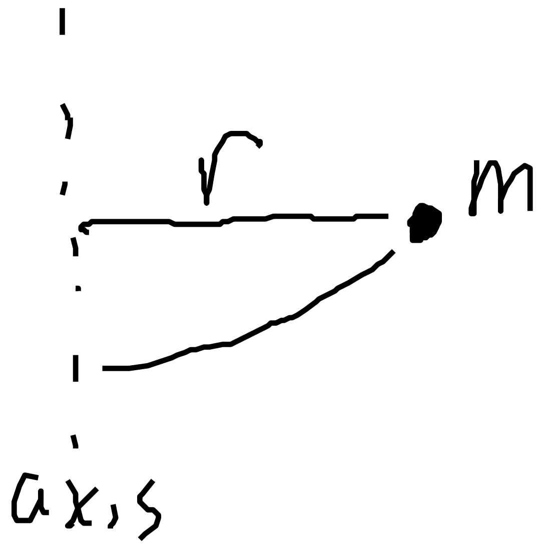
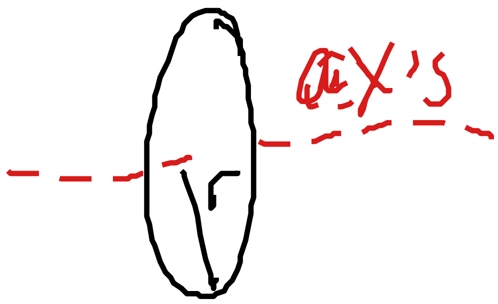
Point mass



$I = mr^2$

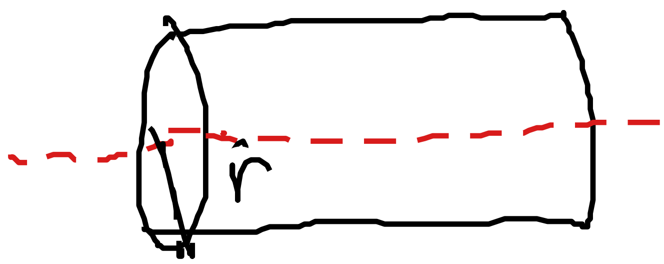
$kg \cdot m^3$

Ring



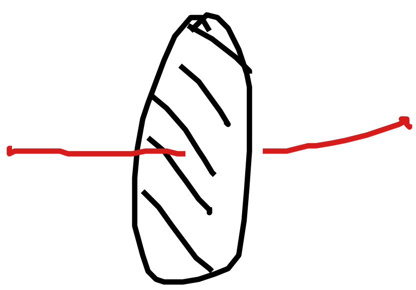
$I = Mr^2$

also for hollow cylinder



$I = Mr^2$

Disk



$I = \frac{1}{2}Mr^2$

For round things rotating around central axis

$$I = cMR^2$$

shape constant

mass

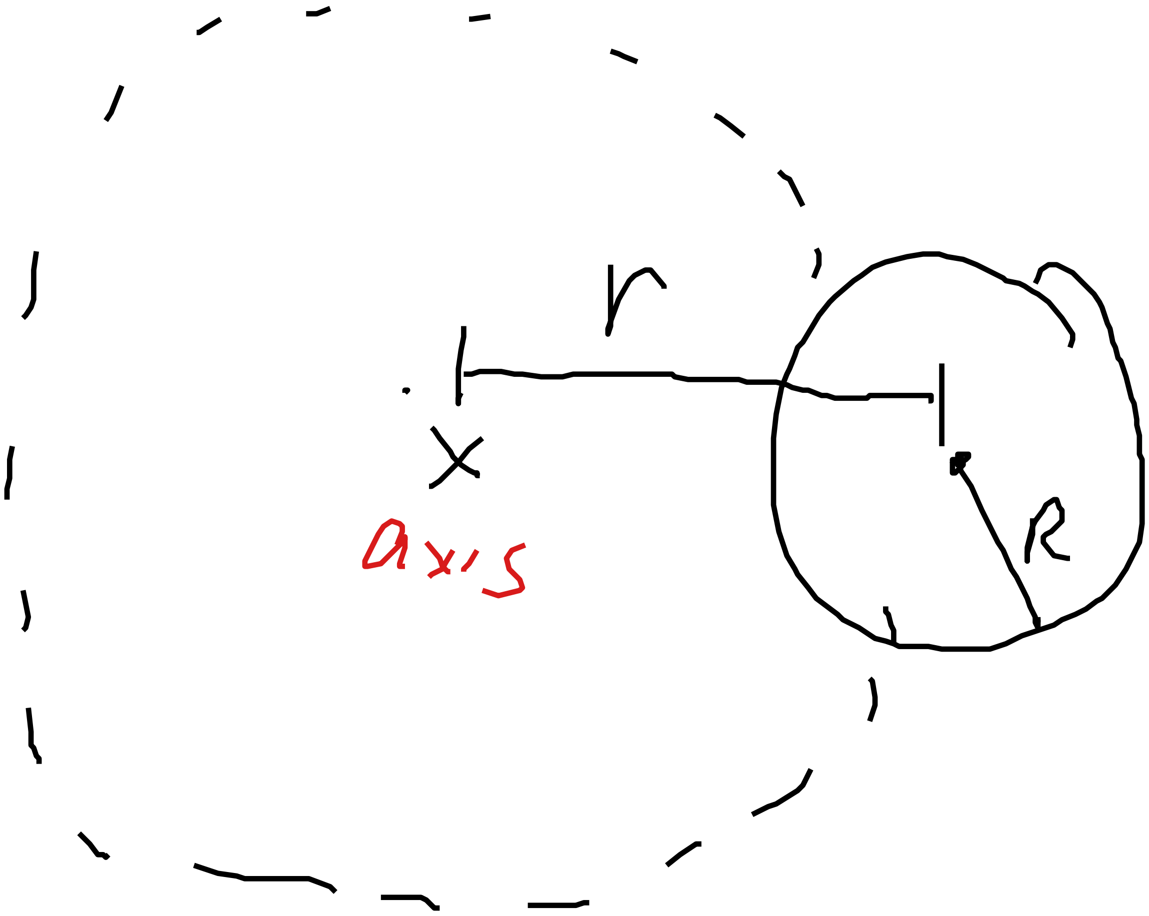
radius

solid sphere $\qquad c = \frac{2}{5}$

hollow sphere $\qquad c = \frac{2}{3}$

generic wheel $\qquad$ c could be anything

$$I = \frac{1}{2} M R^2$$



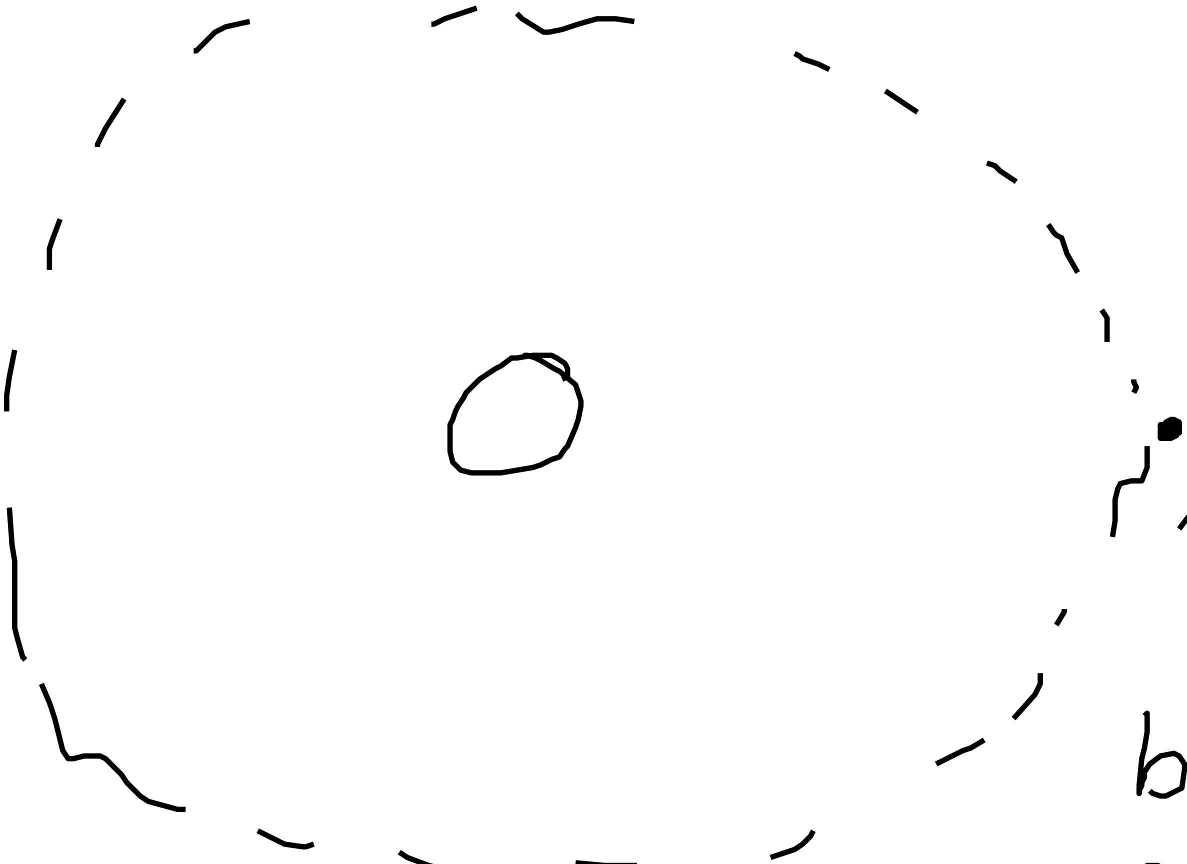
$$I = \frac{1}{2} M R^2 + \underbrace{M r^2}_{\text{parallel-axis theorem}}$$

r: distance from axis to central axis.

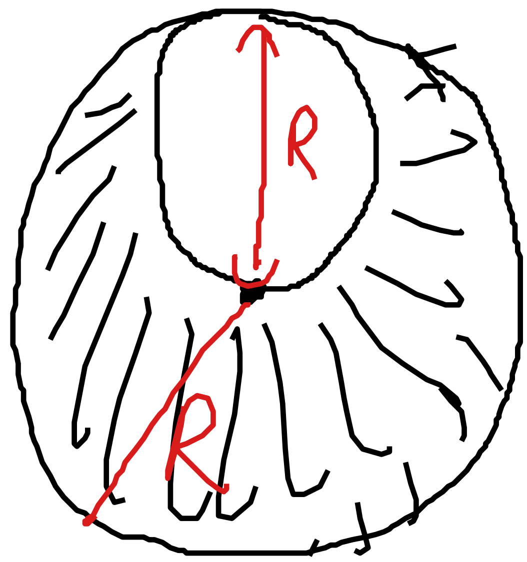
If $r \gg R$

$I \simeq Mr^2$

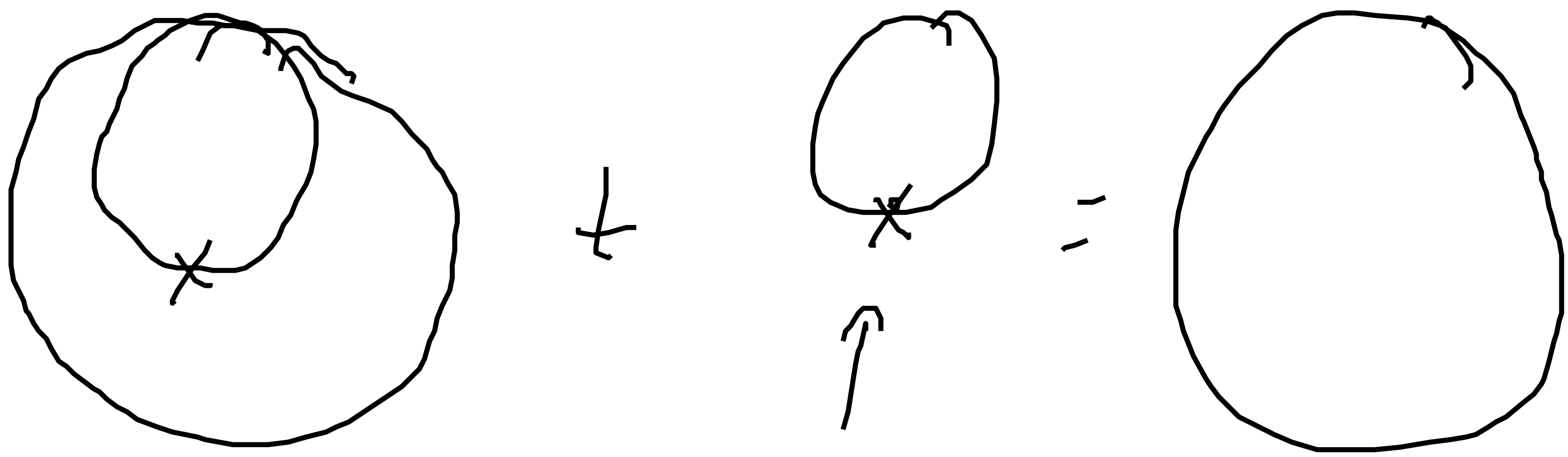
same as for a point



basically a point if $r \gg R$



What is moment of inertia?



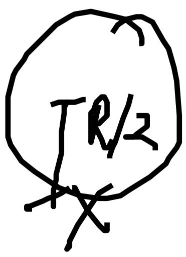
$$I + I_{hole} = \frac{1}{2}MR^2$$

$$I = \frac{1}{2}MR^2 - I_{hole}$$

(or cutout)

$$I_{hole} = \frac{1}{2} M_{hole} \left(\frac{R}{2}\right)^2 + \underbrace{M_{hole} \left(\frac{R}{2}\right)^2}_{m r_{shift}^2}$$

Parallel-axis theorem



$$= \frac{3}{2} M_{hole} \left(\frac{R}{2}\right)^2$$

$$= \frac{3}{8} M_{hole} R^2$$

Then it got complicated. –

$$\frac{M_{hole}}{M_{disk}} = \frac{A_{hole}}{A_{disk}} = \frac{\pi (R/2)^2}{\pi R^2} = \frac{1}{4}$$

$$M_{hole} = \frac{1}{4} M_{disk}$$

$$M + M_{hole} = M_{disk}$$

$$M + \frac{1}{4} M_d = M_d \rightarrow M = \frac{3}{4} M_d$$

$$M_d = \frac{4}{3} M$$

$$M_{hole} = \frac{1}{4} M_d = \frac{1}{3} M$$

$$I_{hole} = \frac{3}{8}\left(\frac{1}{3} M\right) R^2$$

$$= \frac{1}{8} M R^2$$

$$I = \frac{1}{2} M R^2 - \frac{1}{8} M R^2$$

$$= \boxed{\frac{3}{8} M R^2}$$

Whew!

Kinetic energy of a rotating object

$$E_k = \frac{1}{2} I \omega^2$$

compare

$$E_k = \frac{1}{2} m v^2$$

For a rolling object: (without slipping)

$v = r\omega$

$$E_k = E_{k,\text{moving}} + E_{k,\text{rotating}}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(cmr^2)\left(\frac{v}{r}\right)^2$$

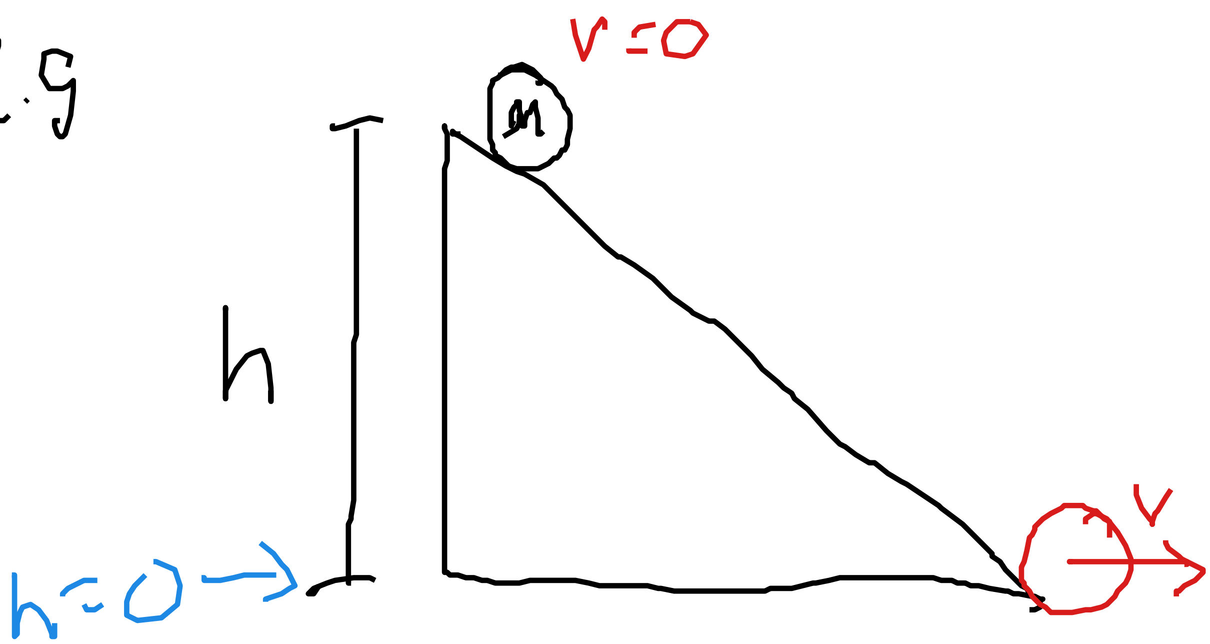
$$= \frac{1}{2}mv^2 + \frac{1}{2}cm\cancel{r^2}\frac{v^2}{\cancel{r^2}}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}cmv^2$$

Rolling object without slipping

$$\boxed{E_k = \frac{1}{2}(1+c)mv^2}$$

e.g



Initial energy is just gravitational

$$E_i = mgh$$

Final energy

$$E_f = \frac{1}{2}(1+c)mv^2$$

If no friction or energy loss,

$$\cancel{m}gh = \frac{1}{2}(1+c)\cancel{m}v^2$$

$$v^2 = \frac{2gh}{1+c} \rightarrow v = \sqrt{\frac{2gh}{1+c}}$$

No rolling: $c = 0$

$$v = \sqrt{2gh}$$

As c increases, v decreases

— some of the initial energy goes into spinning the object and not increasing its speed

solid sphere: $c = \frac{2}{5} = 0.4$

solid cylinder: $c = 1/2 = 0.5$

hollow sphere: $c = 2/3 = 0.67$

hollow cylinder: $c = 1$

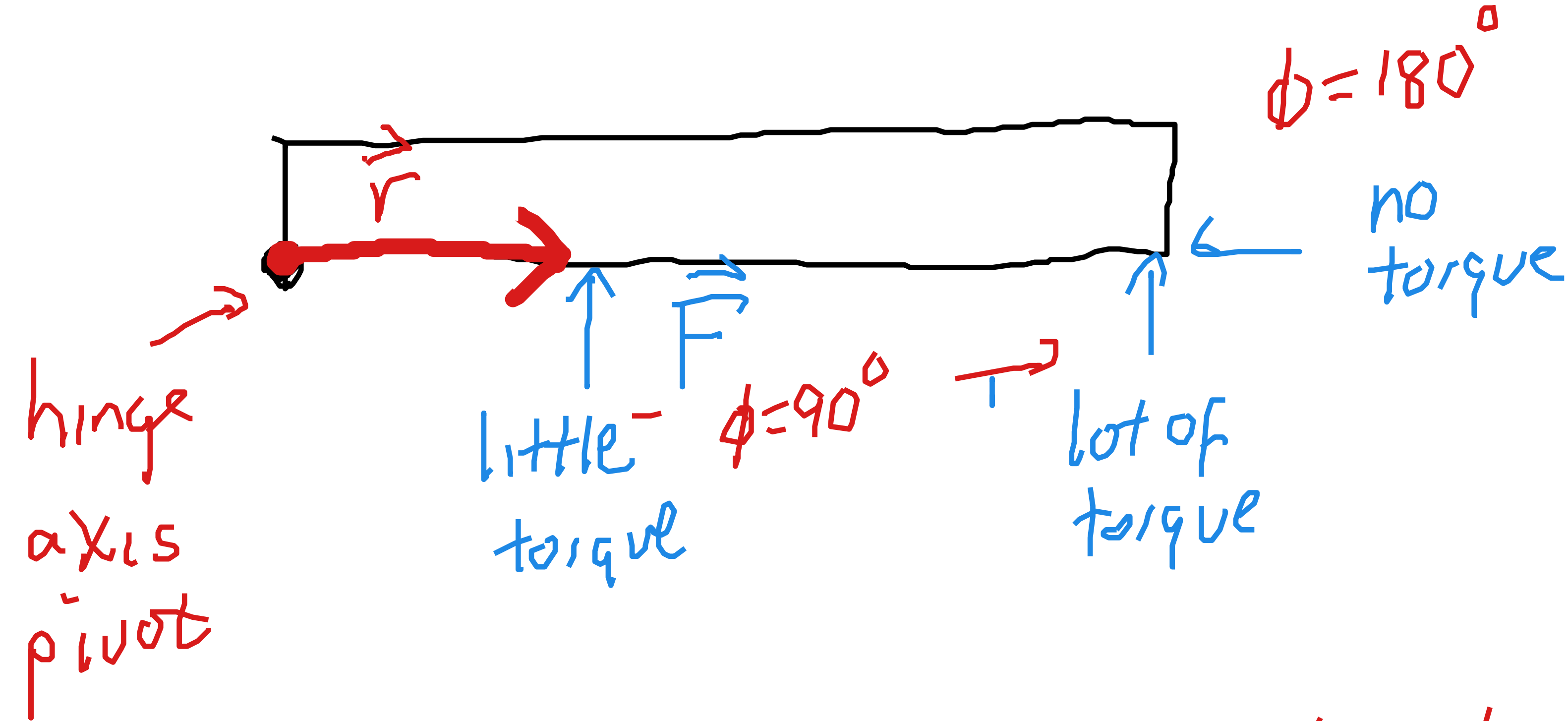
slower

Angular analog to force:

torque

tendency of a force to make an object spin

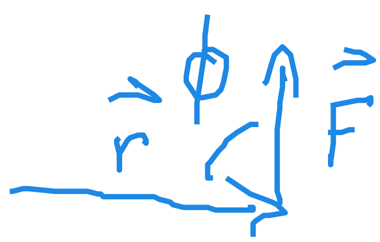
- depends not just on magnitude of force but direction & location of force relative to the axis



$\vec r$: vector from axis to application of force

lower-case tau

$$\tau = |\vec r|\,|\vec F| \sin\phi$$

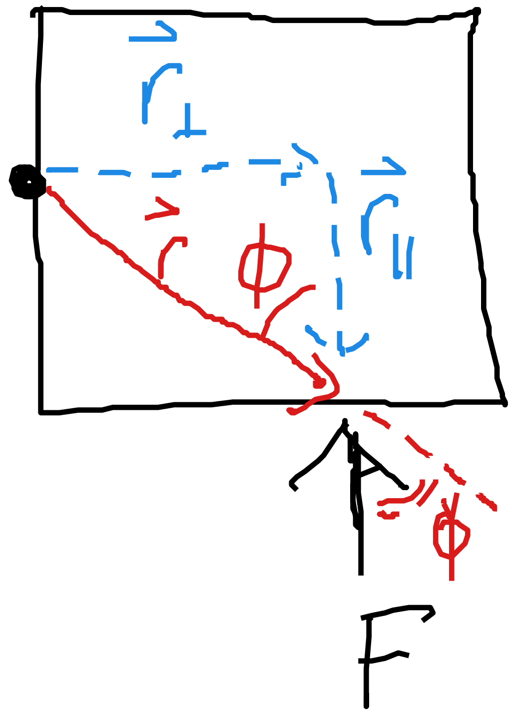


if ϕ is 90° $(\vec r \perp \vec F)$

$$\tau = rF$$

if ϕ is 0° or 180°, $(\vec r \parallel \vec F)$

$$\tau = 0$$



$\vec{r}_\perp$ = component of $\vec{r}$ $\perp$ to $\vec{F}$

$r_\parallel$ similarly

$$r_\perp = r \sin\phi$$

$$\tau = r_\perp F$$

$r_\perp$: "moment arm"

$$\tau = I\alpha \qquad F = ma$$

$$W = \tau \Delta\theta \qquad W = F\Delta x$$

work

$$= Fd$$

Examples

- A large stone wheel ($m = 40\,kg$, $R = 0.5\,m$) is spinning around, making 8 turns in 2 minutes. It has some axial friction applying a torque of $\tau = -0.2\,Nm$. How long until it stops, and how many times does it spin around?

$\Delta\theta$ NEED

ω_i $\frac{8 \text{ rev}}{2 \text{ min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60 \text{ s}} = 0.42 \text{ rad/s}$

ω_f 0 (stops)

α $\tau/I = -\frac{0.2 \text{ Nm}}{\frac{1}{2} m R^2} = -\frac{0.2}{\frac{1}{2}(40)(0.5)^2}$

Δt : NEED $= -0.04 \frac{\text{rad}}{\text{s}^2}$

$$\omega_f = \omega_i + \alpha \Delta t$$

$$0 = 0.42 - 0.04 \Delta t$$

$$\Delta t = \frac{0.42}{0.04} = 10.5 \text{ s}$$

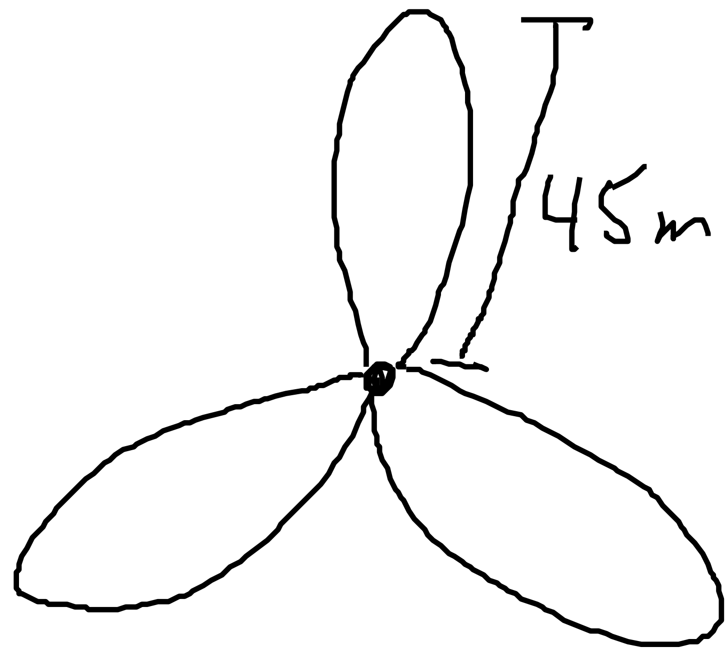
$$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)\Delta t$$

$$= \frac{1}{2}(0.42 + 0)(10.5) = 2.21 \text{ rad}$$

$$2.21\,\cancel{\text{rad}} \times \frac{1\text{ rev}}{2\pi\,\cancel{\text{rad}}}$$

$$= \boxed{0.35\text{ rev}}$$

Windmill



Windmill spins 15rpm and generates 3MW of power What is the minimum torque of wind on the windmill?

$$\frac{W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t}$$

$$\boxed{P = \tau\omega}$$

$P = Fv$

$$P \geq 3\,MW = 3\times10^6\,W$$

$$\omega = 15\,\frac{rev}{min} \times \frac{1\,min}{60\,s} \times \frac{2\pi\,rad}{1\,rev}$$

$$= 1.57\,rad/s$$

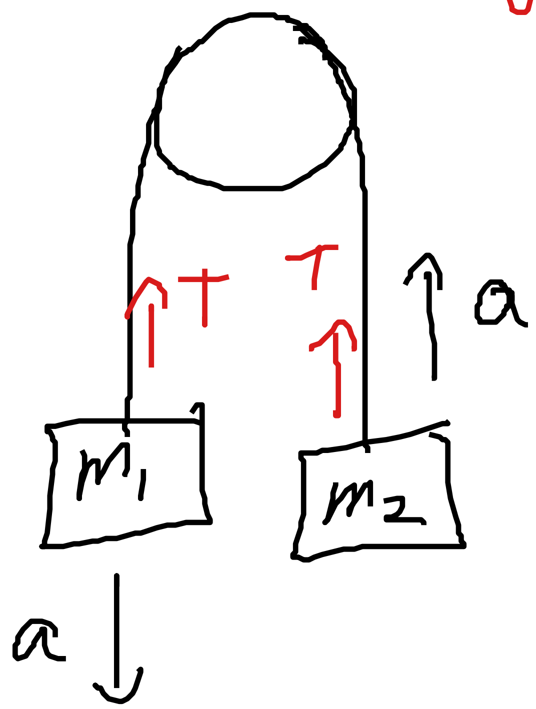
$$\tau = \frac{P}{\omega} \geq \frac{3\times10^6\,W}{1.57\,rad/s}$$

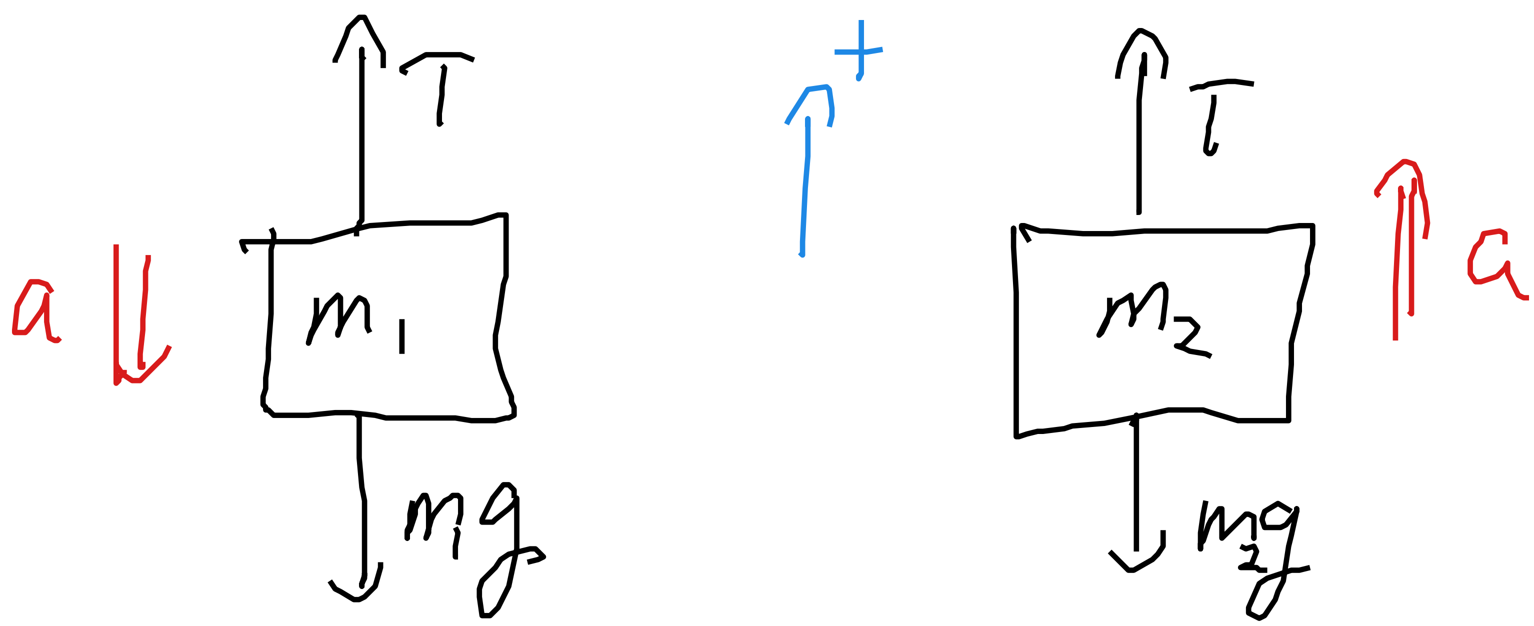
$$\tau \geq 1.9\times10^6\,Nm$$

Pulleys

massless, frictionless pulley
(just changes direction
of tension)

$m_1 > m_2$





$F_{net} = ma$

$T - mg = -m_1 a$

$F_{net} = ma$

$T - m_2 g = +m_2 a$

- solve m_1 equation for T

$$T = m_1 g - m_1 a$$

- substitute into m_2 equation

$$T - m_2 g = +m_2 a$$

$$(m_1 g - m_1 a) - m_2 g = m_2 a$$

$$(m_1 - m_2) g = (m_1 + m_2) a$$

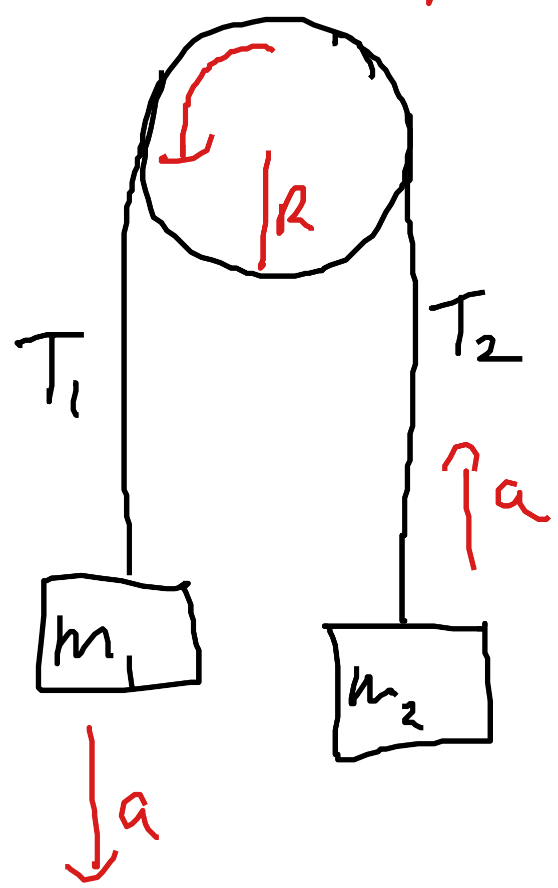
$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Suppose the pulley has mass

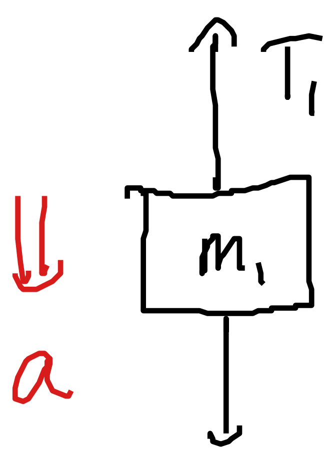
no slippage: $\Delta y = R\Delta\theta$

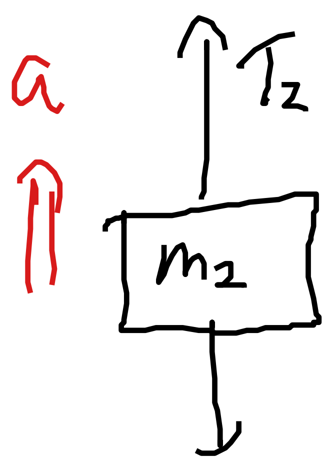
$v = R\omega$

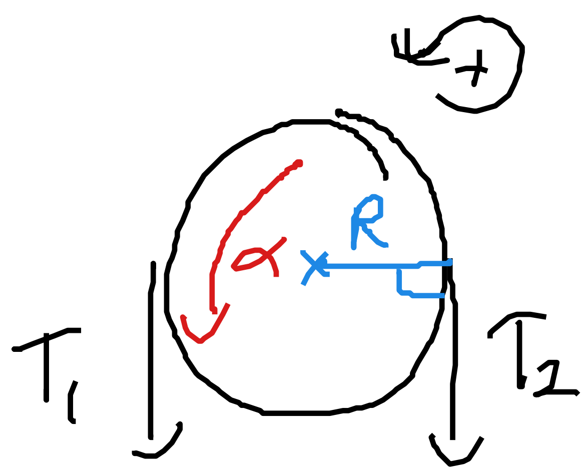
$a = R\alpha$



$\uparrow +$







$\tau_1 = +RT_1$ $\quad$ $\tau_2 = -RT_2$

$T_1 - m_1 g = -m_1 a$ $\quad$ $T_2 - m_2 g = +m_2 a$

$T_1 = m_1 g - m_1 a$ $\quad$ $T_2 = m_2 g + m_2 a$

on pulley

$$\tau_{net} = +RT_1 - RT_2$$

$$= R(T_1 - T_2)$$

$$\tau_{net} = I\alpha$$

$$\cancel{R}(T_1 - T_2) = (cM\cancel{R}^2)\left(\frac{a}{\cancel{R}}\right)$$

$$T_1 - T_2 = cMa$$

$$(m_1 g - m_1 a) - (m_2 g + m_2 a) = cMa$$

$$m_1 g - m_1 a - m_2 g - m_2 a = cMa$$

$$m_1 g - m_2 g = (m_1 + m_2 + cM)a$$

$$a = \frac{m_1 - m_2}{m_1 + m_2 + cM} g$$