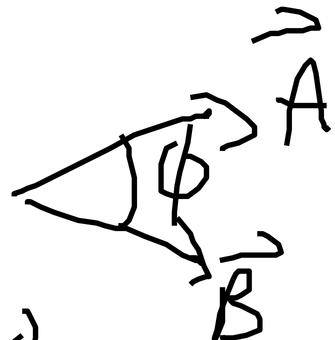
Two ways to multiply two vectors

~~$\vec{A}\,\vec{B}$~~

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\cos\phi$$

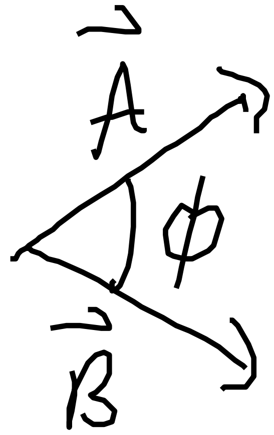
- dot product
- scalar
- 0 when $\phi = 90^\circ$
- AB when $\phi = 0^\circ$



Cross - Product

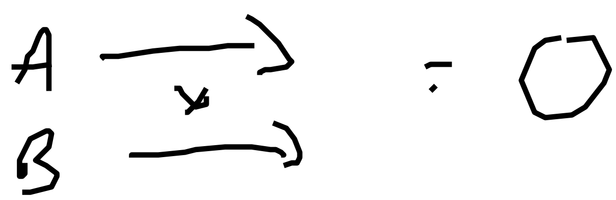
- vector

$\vec{A} \times \vec{B}$

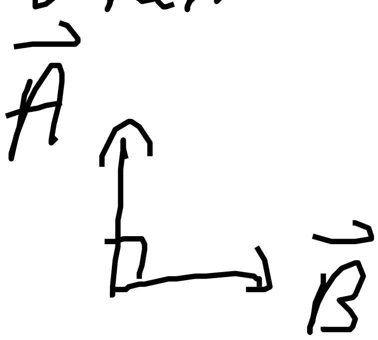


$$|\vec{A} \times \vec{B}| = A B \sin\phi$$

- if $\phi = 0$ or 180°, $A \times B = 0$



- maximal when $\phi = 90^\circ$



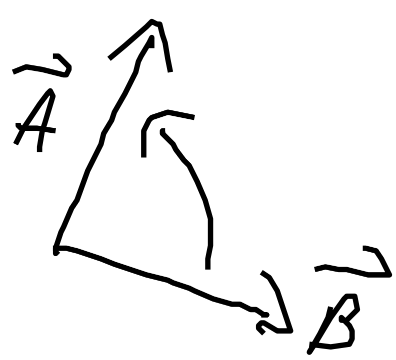
$$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}|$$

Direction of $\vec{A} \times \vec{B}$ is perpendicular to $\vec{A}$ & $\vec{B}$ & the plane they span

$\vec{A} \times \vec{B}$ into the page

$\otimes$

$\vec{B} \times \vec{A}$



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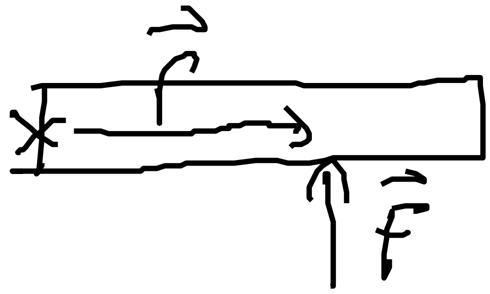
$\odot$

$$\vec{A} \times \vec{B} = - \vec{B} \times \vec{A}$$

non-commutative

$$\tau = rF\sin\phi$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\vec{r}\,\rightarrow \;\times\; \uparrow\vec{F} \;=\; \tau \odot$$

out of page

$$\tau = r_\perp F = r F_\perp$$

$$|\vec{A} \times \vec{B}| = A_\perp B = A B_\perp$$

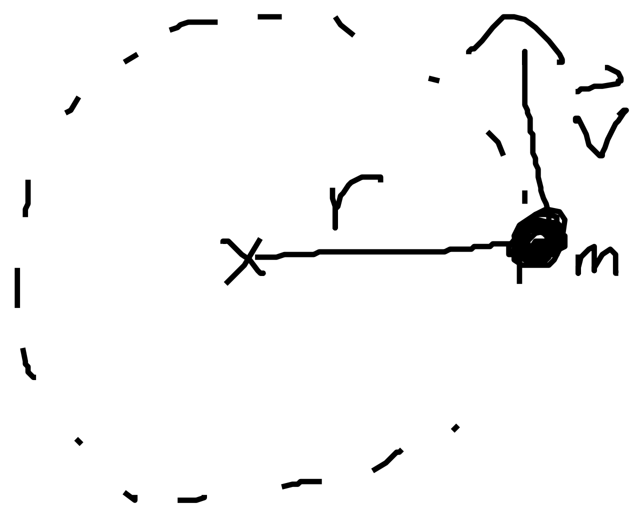
Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$\vec{r}$: vector from pivot

$\vec{p}$: momentum

e.g.



$$\vec{p} = m\vec{v}$$

$$\vec{L} = r \rightarrow \times \; mv \uparrow$$

$$= mvr \; \odot$$

$v = r\omega$

$\vec{L} = m(r\omega)r \odot$

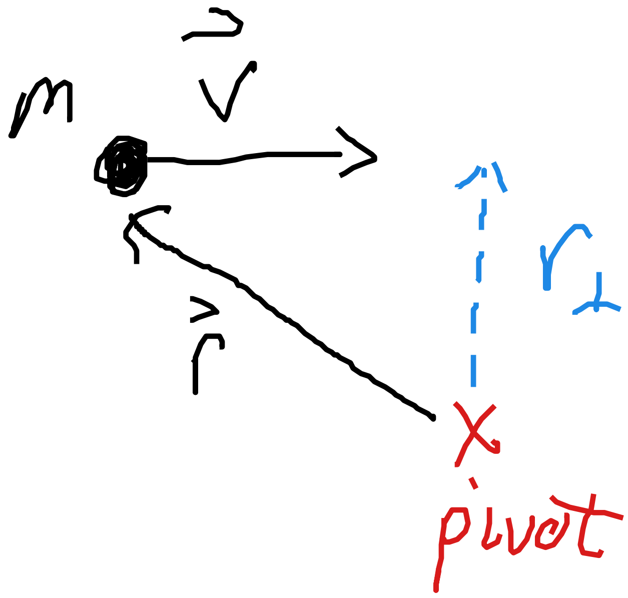
$= \underbrace{mr^2}\omega \odot$

$\vec{L} = I\omega \odot$

$$\boxed{\vec{L} = I\vec{\omega}}$$

Compare $\vec{p} = m\vec{v}$

Object moving in a straight line



$$\vec{L} = \vec{r} \times m\vec{v}$$

$$= m(\vec{r} \times \vec{v})$$

$$= m r_\perp v \quad \otimes$$

$r_\perp$ is unchanged if mass keeps moving to the right

Conservation of Angular Momentum

$$0 \stackrel{?}{=} \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\vec{v} \times (m\vec{v})$$

$$m(\vec{v} \times \vec{v})$$

$$= 0$$

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

$$= m\frac{d\vec{v}}{dt} = m\vec{a}$$

$$= \vec{F}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

more accurate version of N2L

$$\vec{F}\,dt = d\vec{p}$$

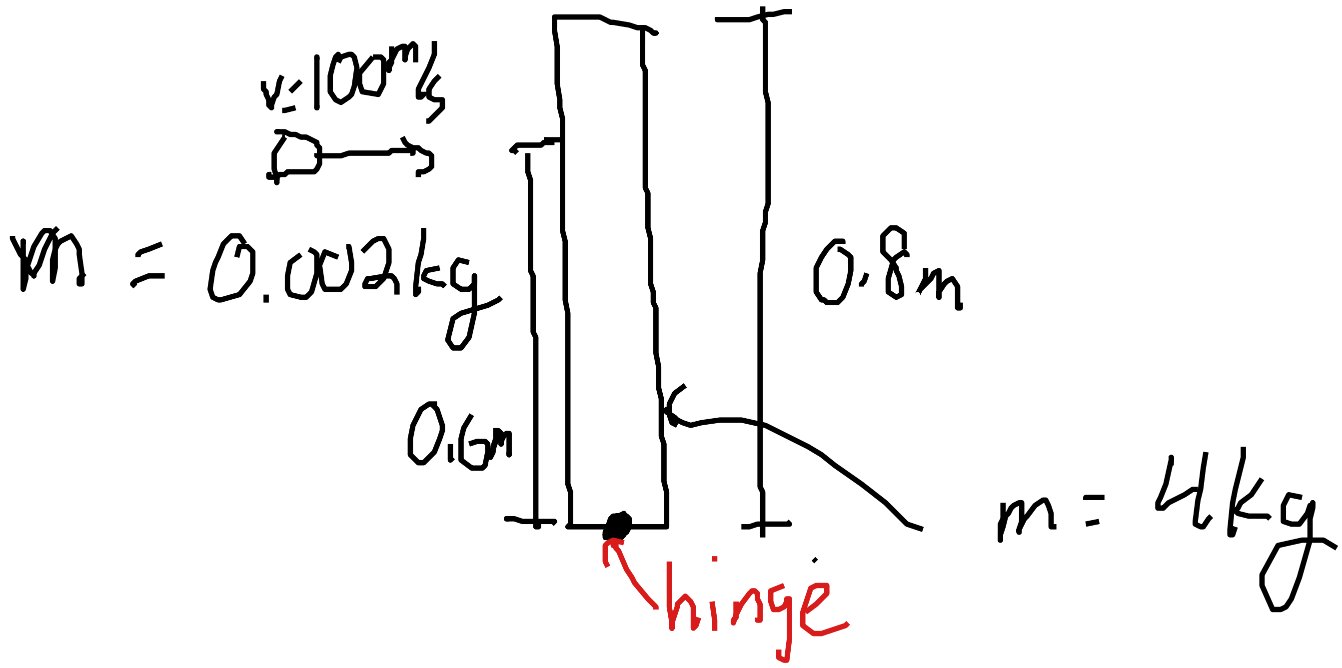
$$\vec{J} = \Delta\vec{p}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

if there is no external torque on a system, then

$$\frac{d\vec{L}}{dt} = 0 \longrightarrow \vec{L} \text{ conserved}$$



Bullet hits the stick and causes it to spin around its hinge. How fast will they spin?

Bullet-stick system feel no external torque around hinge.

$$\vec{L}_i = \vec{L}_f$$

$$\vec{L}_i = \vec{L}_{i,bullet} + \vec{L}_{i,stick}$$

$$m_B V_B (0.6) \otimes + 0$$

$$\vec{L}_f = I \vec{\omega}$$ ← use this formula for spinning extended objects

$$= (I_B + I_s)\omega \otimes$$

$$= \left(m_B (0.6)^2 + \frac{1}{3} m_s (0.8)^2\right)\omega \otimes$$

$m_B = 0.002\,kg$ $V_B = 100\,m/s$

$m_S = 4\,kg$

$$m_B V_B (0.6) = (0.36 m_B + 0.213 m_s)\,\omega$$

$$\omega = \frac{0.6\, m_B V_B}{0.36 m_B + 0.213 m_s}$$

$$= \frac{0.6(0.002)(100)}{0.36(0.002) + 0.213(4)}$$

$$= 0.141 \text{ rad/s}$$

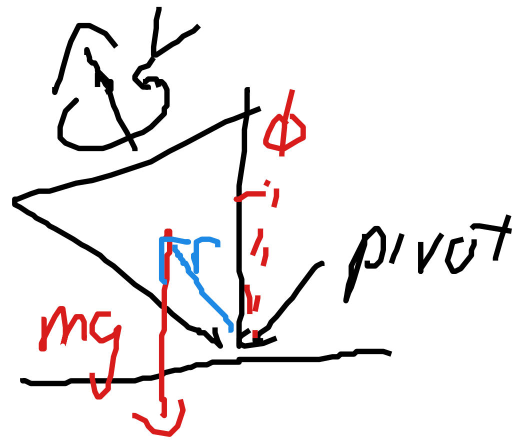
Precession

- top has angular momentum

$$\vec{L} = I\vec{\omega}$$

which points out of its stem

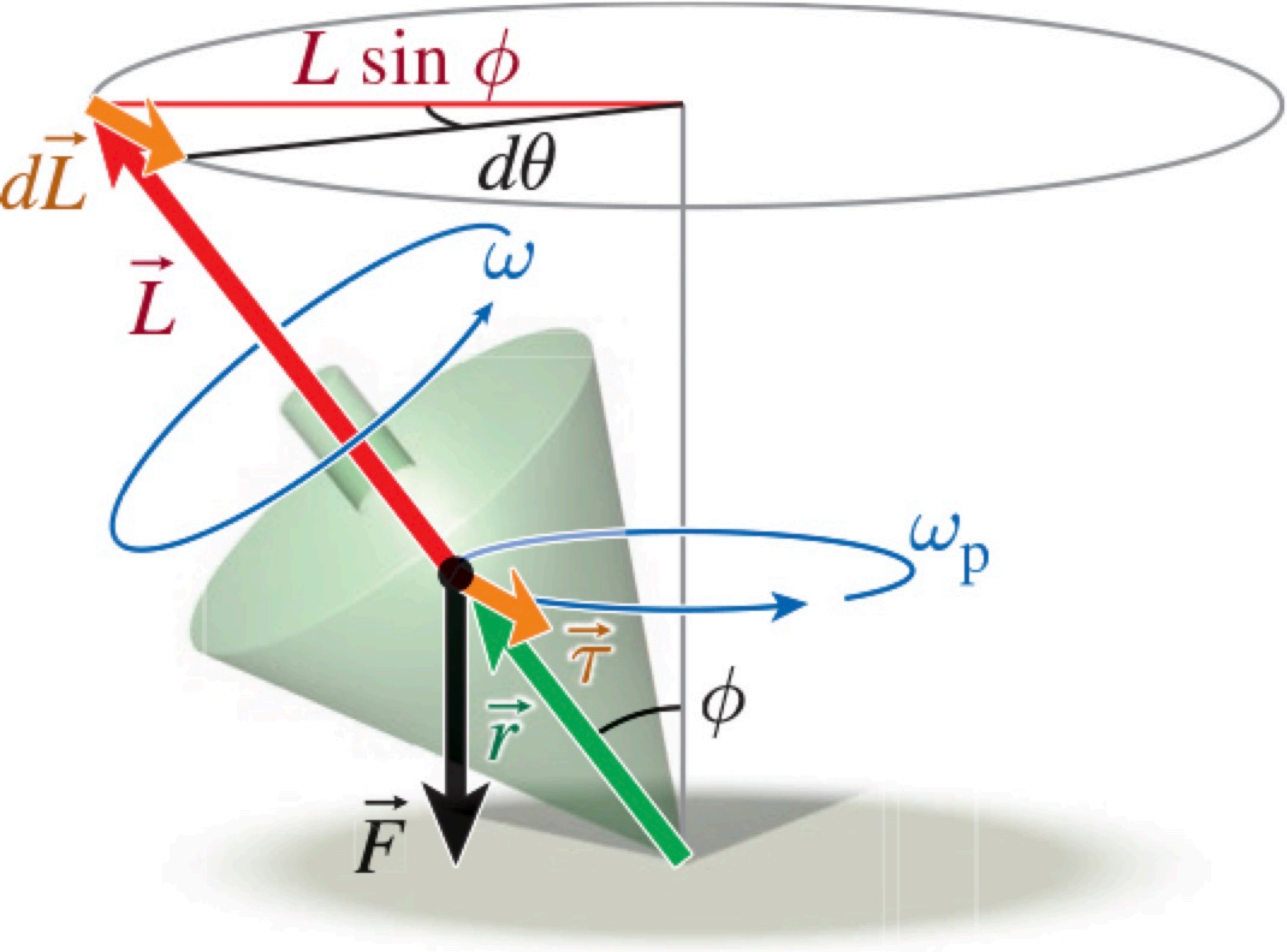
- put pivot at tip of the top



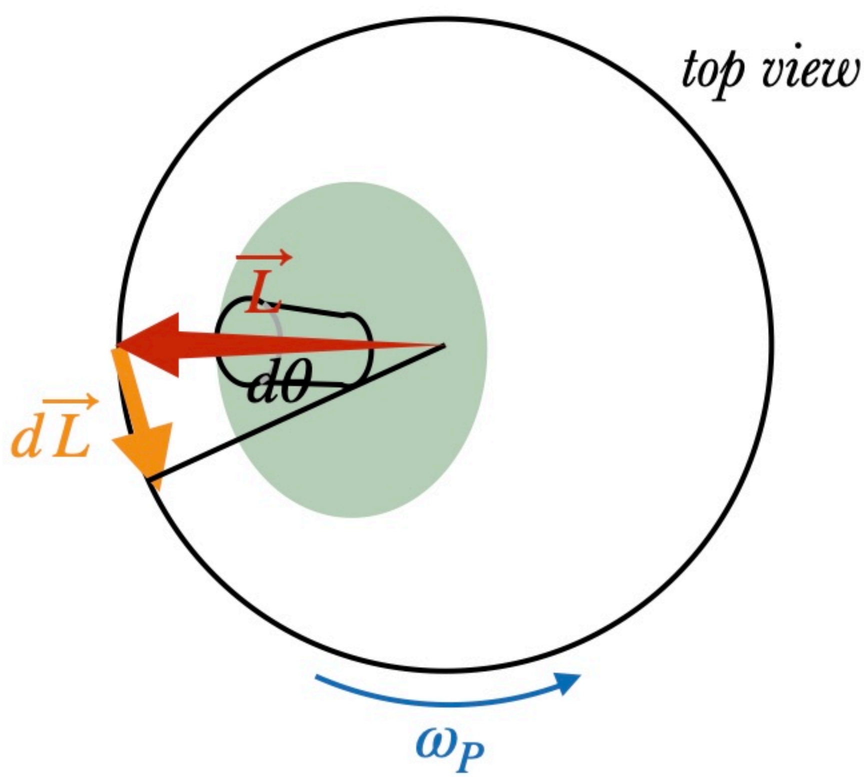
- gravity exerts a force from COM – and also a torque

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\phi$$

points out of the page



$L \sin \phi$
$d\theta$
$d\vec{L}$
$\vec{L}$
ω
ω_p
$\vec{\tau}$
$\vec{r}$
ϕ
$\vec{F}$



top view
$\vec{L}$
$d\theta$
$d\vec{L}$
ω_P

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$d\vec{L} = \vec{\tau}\, dt$$

This causes the angular momentum to spin at a rate ω_p around a vertical axis

In time dt, $\vec{L}$ moves a distance $d\vec{L}$ along a circle with radius $L \sin\phi$

$$d\vec{L} = (L \sin\phi)\, d\theta$$

$$\frac{\vec{\tau}\, dt}{dt} = \frac{L \sin\phi \, d\theta}{dt}$$

$$\tau = L \sin\phi \frac{d\theta}{dt}$$

$$r F \sin\phi = L \sin\phi \, \omega_p$$

$$r mg \sin\phi = L \sin\phi \, \omega_p$$

$$\omega_p = \frac{rmg}{L}$$

$$\omega_p = \frac{rmg}{I\omega}$$

rate of precession
of a top

$$\omega_p = \frac{rmg}{I\omega}$$

r: distance of COM above tip
m: mass of top
g = 9.8
I: moment of inertia of top
ω: rate of spin of top