Center of Mass

$m_1 \qquad m_2 \qquad m_3$

$x_1 \qquad x_2 \qquad x_3$

$$X_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

if $\quad m_1 = m_2 = m_3 = m$

$$X_{com} = \frac{m x_1 + m x_2 + m x_3}{m + m + m}$$

$$= \frac{x_1 + x_2 + x_3}{3}$$

just normal average

If masses are different we get a <u>weighted</u> <u>average</u> which tends to be closer to the more massive particles

2kg　　　com　6kg

•　　　　x　　•

$x=0$　　　　$x=3$

$$X_{com} = \frac{2(0)+6(3)}{2+6}$$

$$= \frac{18}{8} = 2\tfrac{1}{4}$$

6kg　com　　　2kg

•　　x　　　　•

$x=0$　　　　$x=3$

$$X_{com} = \frac{6(0)+2(3)}{6+2} = \frac{6}{8} = \frac{3}{4}$$

m_1

(x_1, y_1)

m_2

(x_2, y_2)

$$\vec{r}_{com} = \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \right)$$

$$\frac{dx_{com}}{dt} = \frac{d}{dt}\,\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{1}{m_1+m_2}\left[m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt}\right]$$

$$= \underbrace{\frac{1}{m_1+m_2}}\left[\underbrace{m_1 V_1 + m_2 V_2}_{P_1 + P_2}\right]$$

$$V_{com} = \frac{1}{M_{tot}} \times P_{tot}$$

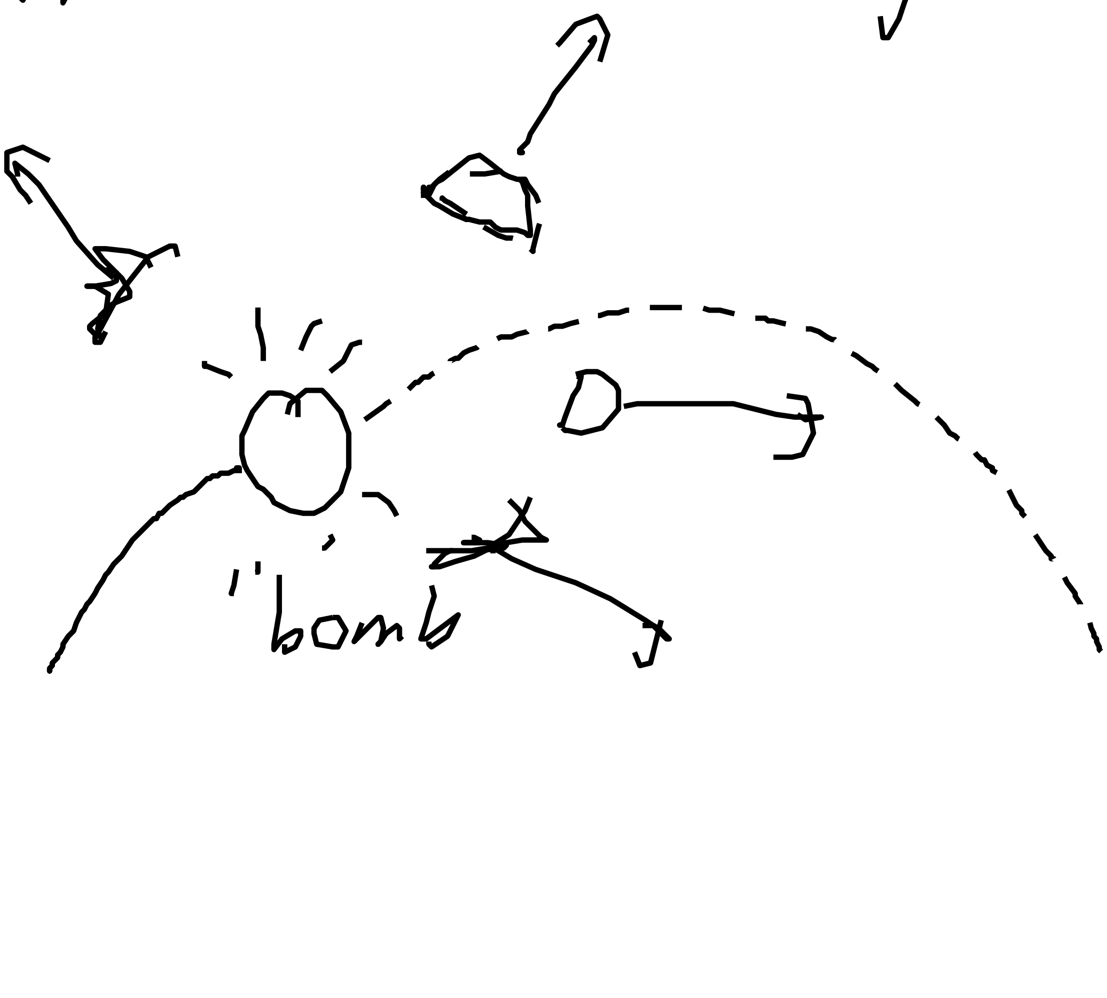
$$\boxed{\vec{P}_{tot} = M_{tot}\,\vec{V}_{com}}$$

When momentum is conserved,

$$\Delta p_{tot} = 0$$

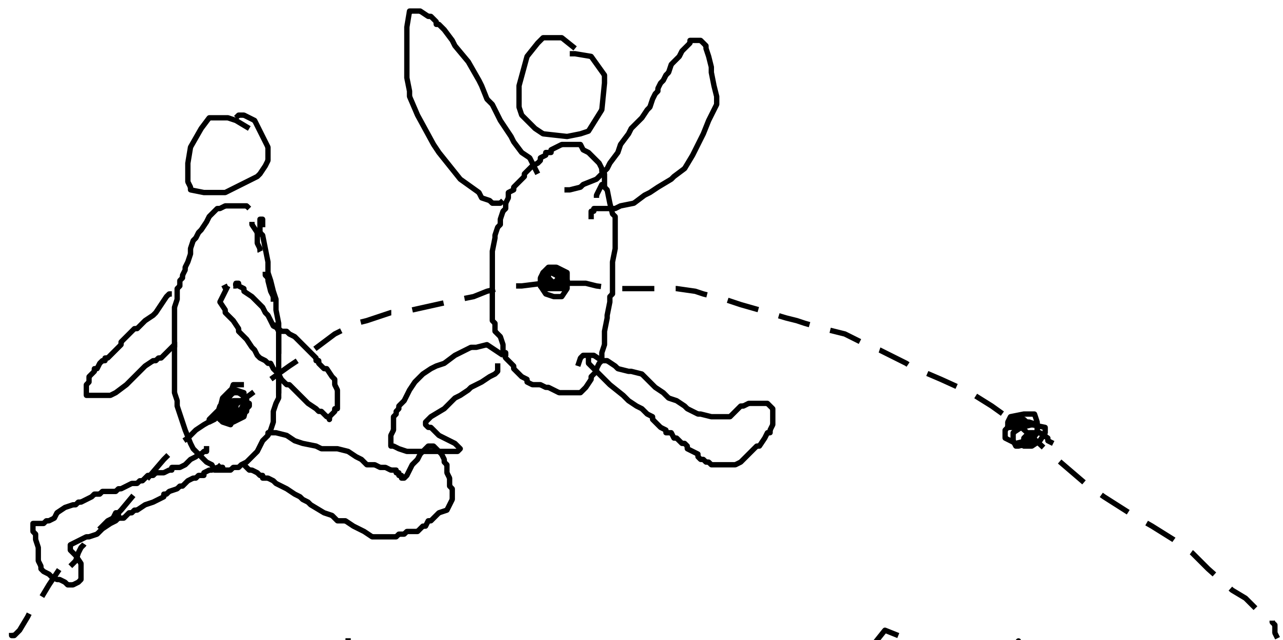
$$\Delta \vec{V}_{com} = 0$$

This can be generalized



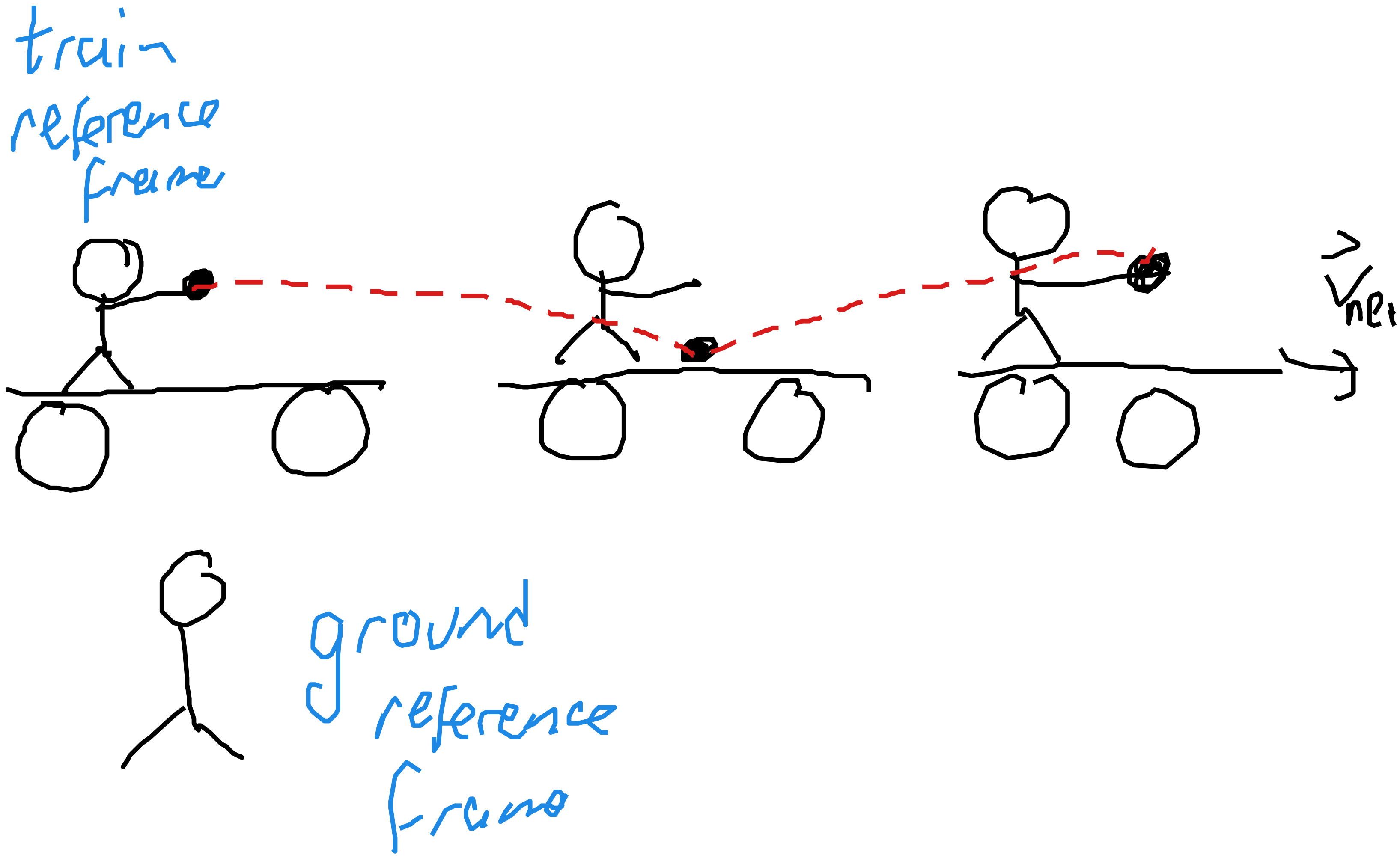
Dancer

Leaps in air



Shift your COM upwards as you reach the top and it looks like your body floats horizontally.

Relative Frames of Reference

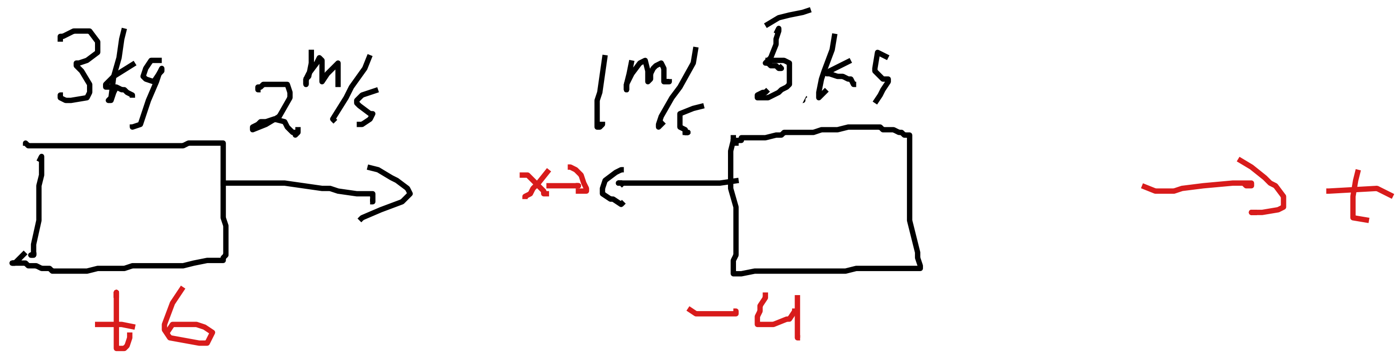


both see the ball obey the laws of physics but in different ways.

To shift to a new frame of reference, $\vec{V}_{new}$

subtract $\vec{V}_{new}$ from all velocities you see

So suppose I take all velocities in a collision problem and subtract V_{com}

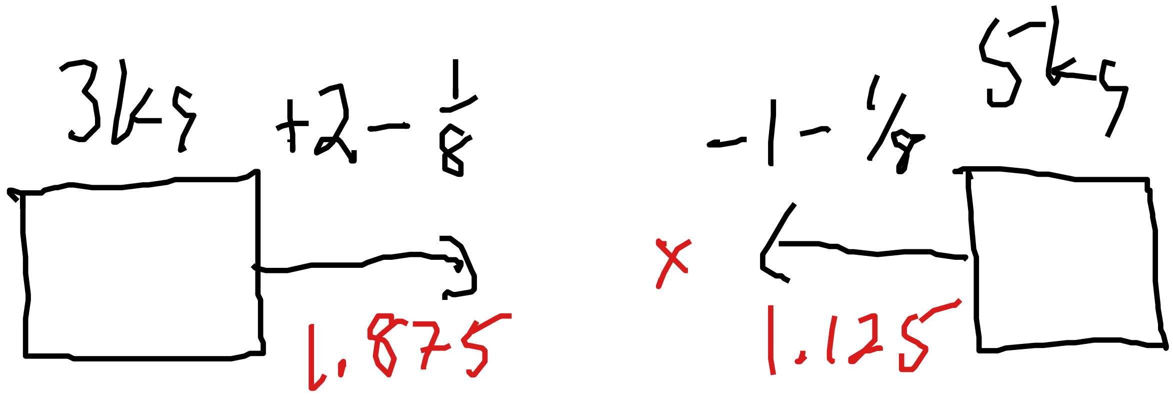


$$\vec{P}_{tot} = +6 - 5 = 1\ Ns$$

$$M_{tot} = 3 + 5 = 8\ kg$$

$$\vec{V}_{com} = \frac{\vec{P}_{tot}}{M_{tot}} = \frac{1\ Ns}{8\ kg} = 0.125\ m/s.$$

if we subtract this velocity from all other velocities,



$p_1 = (3)(1.875)$ $\qquad p_2 =$

$= +5.625 \qquad -5.625$

$p_{tot} = 0$

$P_{tot} = M_{tot} V_{com}$

$0 = M_{tot}\ 0$

After collision,

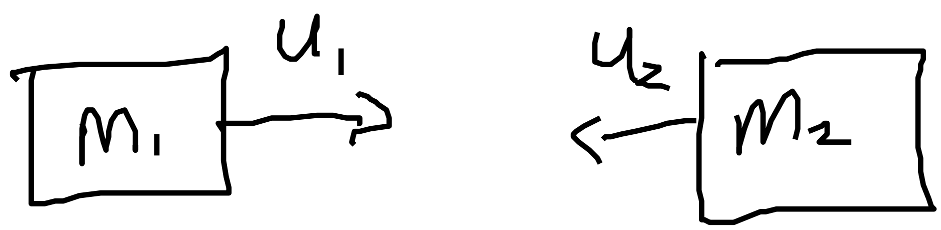
$p_{tot,f}$ is also 0

(doesn't change)

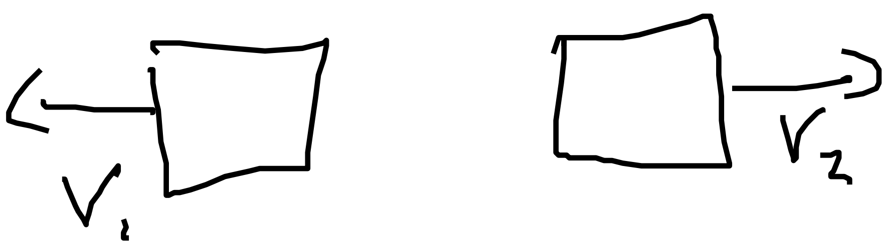
this will make the math easier

WARNING: From this point on the notes are rather a mess.
I would recommend referring to the Collisions PDF

Perfectly elastic collision



u_1 is v_{1i} $\quad$ u_2 is v_{2i}



v_1 is v_{1f} $\quad$ v_2 is v_{2f}

u & v are velocites in the "lab frame" or the observer's frame

u' & v' are velocities in the COM frame

$$\vec{V}_{com} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

$$\begin{aligned}
\vec{u}_1{}' &= \vec{u}_1 - \vec{V}_{com} \\
&= \vec{u}_1 - \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} \\
&= \frac{\vec{u}_1 (m_1 + m_2)}{m_1 + m_2} - \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} \\
&= \frac{1}{m_1 + m_2} \left[\cancel{m_1 \vec{u}_1} + \underline{m_2} \vec{u}_1 - \cancel{m_1 \vec{u}_1} - \underline{m_2} \vec{u}_2 \right]
\end{aligned}$$

$$\boxed{\begin{aligned}
\vec{u}_1{}' &= \frac{m_2}{m_1 + m_2} \left(\vec{u}_1 - \vec{u}_2 \right) \\
\vec{u}_2{}' &= \frac{m_1}{m_1 + m_2} \left(\vec{u}_2 - \vec{u}_1 \right)
\end{aligned}}$$

initial velocities in COM frame.

$$\vec{P}_{ki}' = m_1 \vec{u}_1' + m_2 \vec{u}_2'$$

$$= \frac{m_1 m_2}{m_1+m_2}(\vec{u}_1 - \vec{u}_2) + \frac{m_2 m_1}{m_1+m_2}(\vec{u}_2 - \vec{u}_1)$$

$$= \frac{m_1 m_2}{m_1+m_2}\left[\vec{u}_1 - \vec{u}_2 + \vec{u}_2 - \vec{u}_1\right]$$

$$= 0$$

$$E_{ki} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$= \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 \left|\vec{u}_1' - \vec{u}_2'\right|^2$$

$$+ \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 \left|\vec{u}_1' - \vec{u}_2'\right|^2$$

$$E'_{K_i} = \frac{1}{2}\left|\vec{u}_1' - \vec{u}_2'\right|^2 \left[\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2}\right]$$

$$= \frac{1}{2}\left|\vec{u}_1' - \vec{u}_2'\right|^2 \left[\frac{m_1 m_2 \cancel{(m_2+m_1)}}{(m_1+m_2)^{\cancel{2}}}\right]$$

$$= \frac{1}{2}\,\frac{m_1 m_2}{(m_1+m_2)^2}\left|\vec{u}_1' - \vec{u}_2'\right|^2$$

Initial KE in the lab. frame where u_1 & u_2 are velocities in the COM frame.

Now suppose momentum is conserved and kinetic energy is conserved.

$$\vec{P}_f' = 0$$

$$E_{kf} = \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)^2}\left|\vec{V}_1' - \vec{V}_2'\right|^2$$

if energy is conserved,

$$\left|\vec{u}_1' - \vec{u}_2'\right|^2 = \left|\vec{V}_1' - \vec{V}_2'\right|^2$$

$$\vec{P}_i' = m_1\vec{u}_1' + m_2\vec{u}_2' = 0$$

$$m_1\vec{u}_1' = -m_2\vec{u}_2'$$

$$\vec{u}_2' = -\frac{m_1}{m_2}\vec{u}_1'$$

$$\vec{u}_1 - \vec{u}_2$$

$$= \vec{u}_1 - \left(-\frac{m_1}{m_2}\vec{u}_1'\right)$$

$$= \vec{u}_1 + \frac{m_1}{m_2}\vec{u}_1'$$

$$= \vec{u}_1'\left(1 + \frac{m_1}{m_2}\right)$$

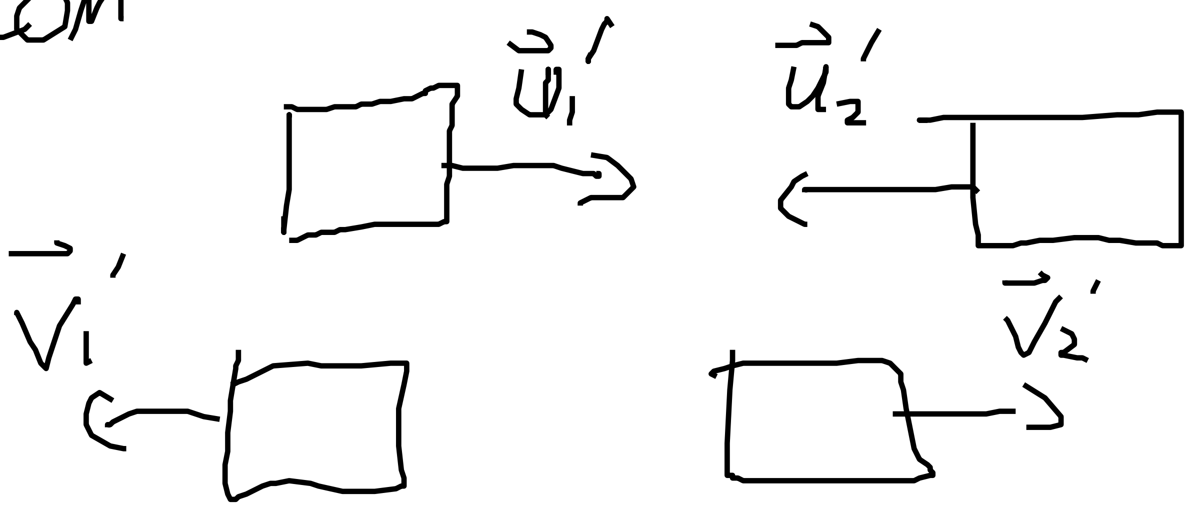
$$|\vec{u}_1' - u_2'|^2 = |\vec{u}_1'|^2\left(1 + \frac{m_1}{m_2}\right)^2$$

$$|\vec{V}_1' - \vec{V}_2|^2 = |\vec{V}_1'|^2\left(1 + \frac{m_1}{m_2}\right)^2$$

$$|u_1'|^2 \cancel{\left(1+\frac{m_1}{m_2}\right)^2} = |v_1'^2| \cancel{\left(1+\frac{m_1}{m_2}\right)^2}$$

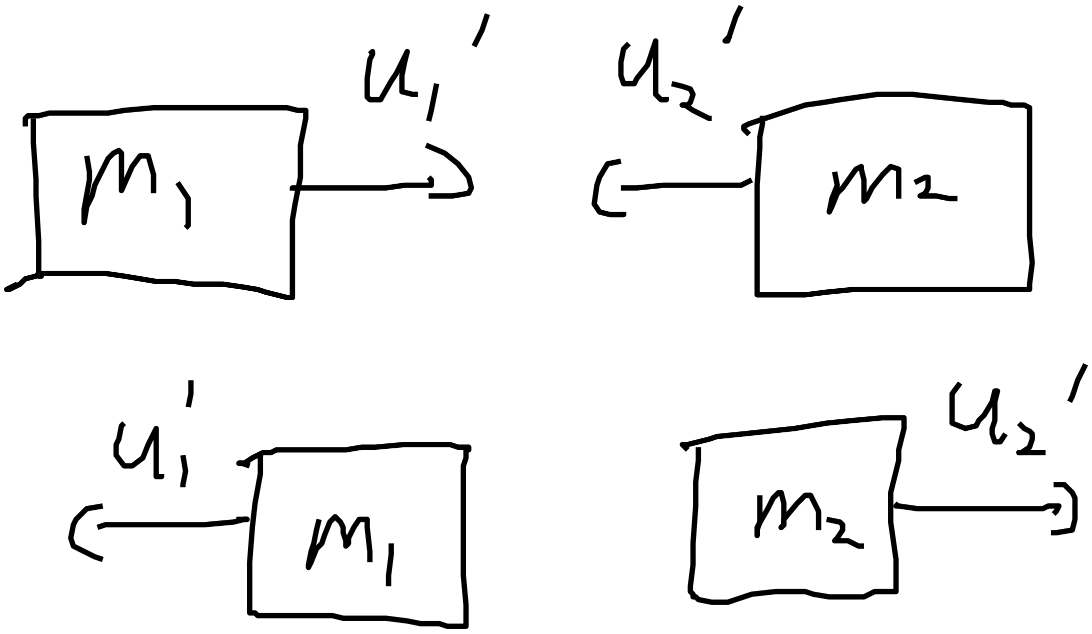
$$|\vec{u}_1'|^2 = |\vec{v}_1'^2|$$

in COM



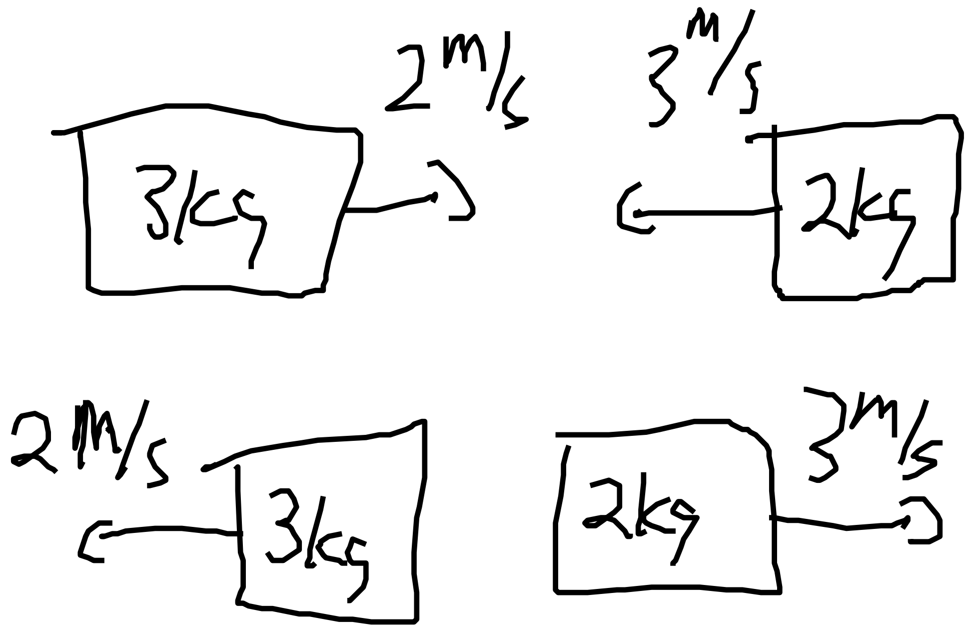
speed of m_1 before the collision = speed after the collision in COM frame

In COM frame



in COM frame

$\vec{p} = +6 - 6 = 0$



elastic collision when $\vec{p}$ & $\vec{E}_k$ are both conserved.

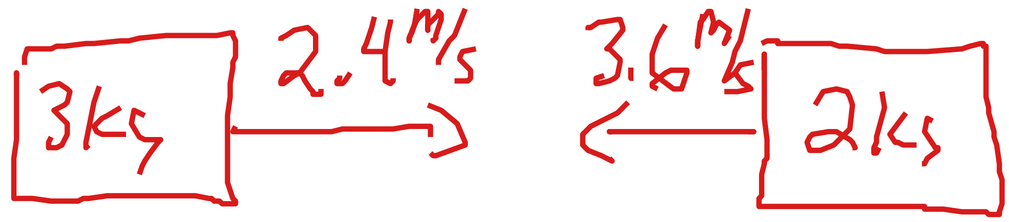
elastic collision

1) find $\vec{V}_{com} = \frac{\vec{P}_{tot}}{M_{tot}}$

e.g. $\frac{3\cdot 2 - 4\cdot 2}{5} = -0.4\ m/s$

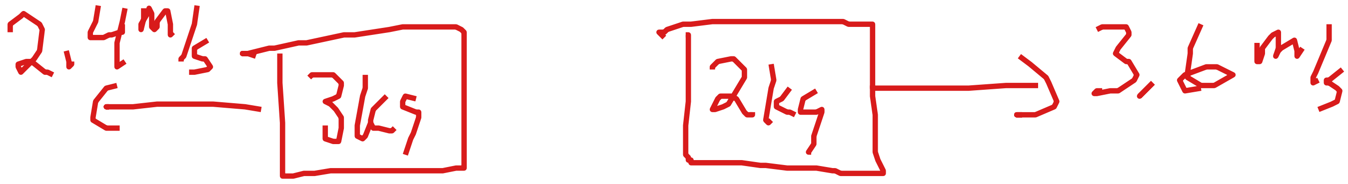
2) Shift all velocities by subtracting V_{com}

e.g.

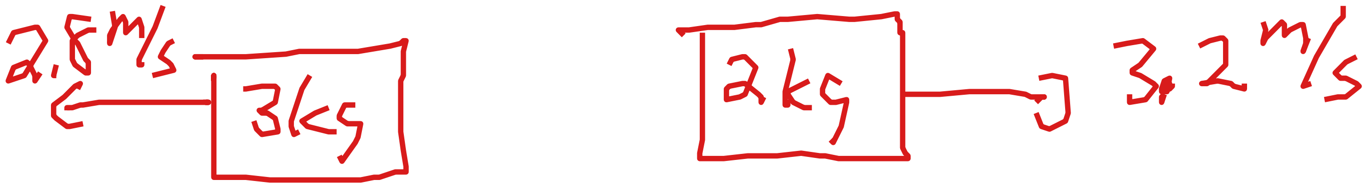


3) Velocities in com flip

e.g.



4) Add $\vec{V}_{com}$ to get to lab frame



Example

1) $\vec{P}_{tot} = +16 - 6 = 10 \text{ Ns}$

$M_{tot} = 10 \text{ kg}$

$V_{com} = \dfrac{\vec{P}_{tot}}{M_{tot}} = \dfrac{10 \text{ Ns}}{10 \text{ kg}} = +1 \text{ m/s}$

2)

3)

4)

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Solving these two equations for an elastic collision simultaneously is <u>really</u> <u>hard</u> — hence the COM detour

$$E_{ki} = \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)^2} |u_1' - u_2'|^2$$

$$E_{kf} = \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)^2} |v_1' - v_2'|^2$$

Macroscopic collisions are never perfectly elastic.

$$e = \frac{E_{kf}}{E_{ki}} = \left| \frac{\vec{v}_1' - \vec{v}_2'}{\vec{u}_1 - \vec{u}_2'} \right|^2$$

coefficient of restitution

(% of energy remaining)

If $e = 1$ (or 100%)

$$|u_1' - u_2'|^2 = |v_1' - v_2'|^2$$

elastic collision

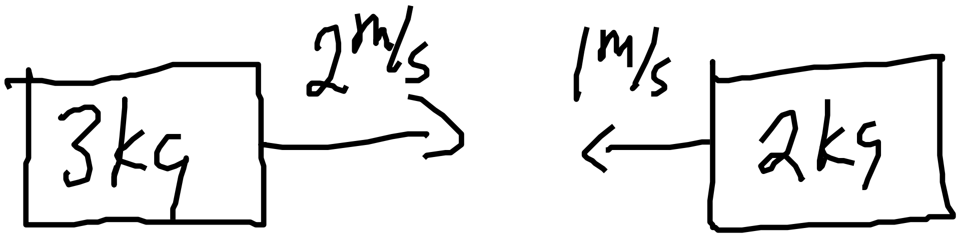
If $e < 1$: inelastic collision

If $e > 1$: superelastic collision (explosion)

If $e = 0$: perfectly inelastic collision

$$|\vec{v}_1' - \vec{v}_2'|^2 = 0 \rightarrow \vec{v}_1' = \vec{v}_2'$$

two objects stick together

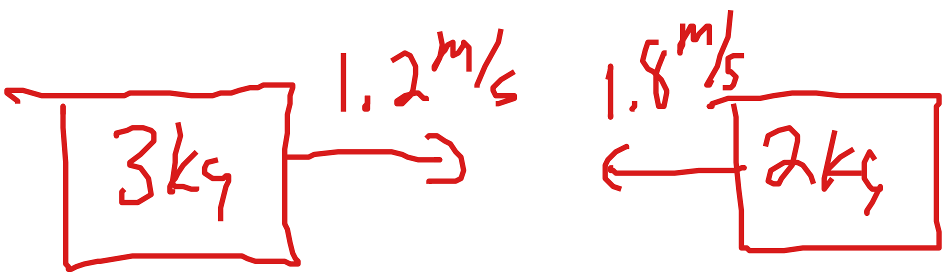


- Convert to COM

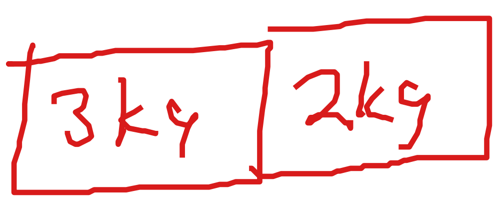
$P_{tot} = +6 - 2 = 4$

$M_{tot} = 5\,kg$

$\vec{V}_{com} = +0.8\,m/s$



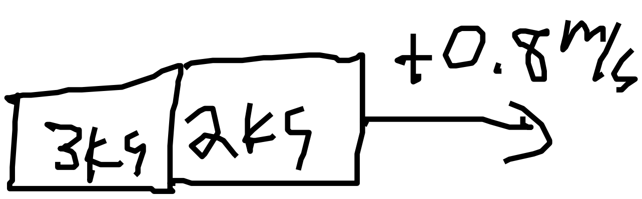
- After perfectly inelastic collision two blocks stick together

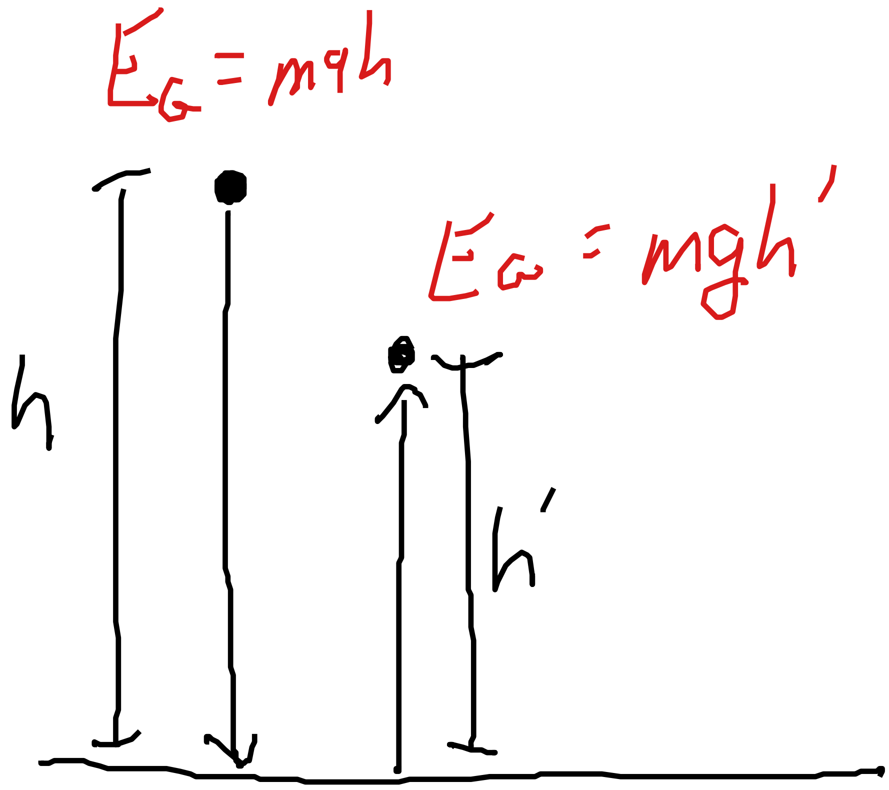


$\vec{P}'_{tot} = 0$,

so $V'_1 = V'_2 = 0$

- Add $\vec{V}_{com}$





if $e = 1$, $h' = h$

if $e = 0$, $h' = 0$

in general,

m

$=$

$$e = \sqrt{\frac{h'}{h}}$$