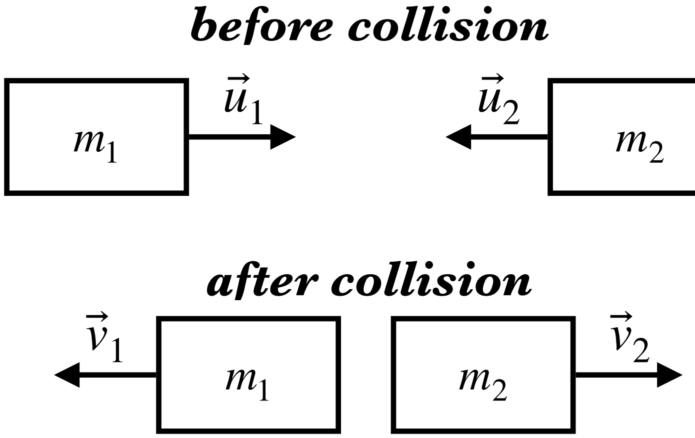
Collisions

Note: I've greyed out mathematical derivations that aren't essential to know.

When two objects collide, the total momentum of the two objects doesn't change before or after the collision. In other words, if



(We're using $\vec{u}$ instead of $\vec{v}_i$ and $\vec{v}$ instead of $\vec{v}_f$ to cut down on subscripts.)

then conservation of momentum says that

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \qquad (1)$$

This is just one equation, which means we can only solve for one unknown, and in fact there are a lot of possible solutions for $\vec{v}_1$ and $\vec{v}_2$. In real life, the way physics chooses the right one is by looking at the kinetic energy before and after the collision, because the kinetic energy is not always conserved.

A *maximally inelastic collision* is the kind where the most kinetic energy is lost; this turns out to be the case when the two objects stick together, and $\vec{v}_1 = \vec{v}_2$. This is a second equation that allows us to solve for both velocities. Note that not *all* the kinetic energy is necessarily lost: because the total momentum must be conserved, the two blocks might continue moving.

At the other extreme is the *elastic collision*, which is when the kinetic energy is completely conserved. This would introduce a second equation

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad (2)$$

We could solve Equations 1 and 2 simultaneously to find v_1 and v_2, but in practice this gets really messy fast, unless we use a trick.

Center of Mass Frame

The center of mass of a set of particles is the weighted average of their positions. For two particles in one dimension, this looks like

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \qquad (3)$$

If these blocks start to move, their center of mass will probably start to move as well. If we take the derivative of both sides of this equation with respect to time, and simplify, we find the velocity of the center of mass:

$$\vec{v}_{com} = \frac{m_1\vec{v}_1 + m\vec{v}_2}{m_1 + m_2} = \frac{\vec{p}_{tot}}{m_{tot}} \qquad (4)$$

and because the momentum is conserved during the collision, the center of mass's velocity is unchanged during the collision as well. If, as an observer, we move with the same velocity as the center of mass, then to us the center of mass remains stationary, and the total momentum of the system is zero to us. Let $\vec{v}'$ be the velocity of a block in the center-of-mass frame, while $\vec{v}$ is the velocity in the "lab frame". Thus

$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0 \implies \vec{v}_2' = -\frac{m_1}{m_2}\vec{v}_1'. \qquad (5)$$

In the center-of-mass frame, the two velocities are always pointing in the opposite direction, and if $m_1 > m_2$ then $v_2' > v_1'$: that is, the more massive block is moving more slowly.

To shift from the lab frame to the center-of-mass frame, we need to subtract the center-of-mass velocity from all velocities in the problem: thus, $\vec{v}' = \vec{v} - \vec{v}_{com}$.

$$v_1' = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{\vec{v}_1(m_1 + m_2) - (m_1 v_1 + m_2 v_2)}{m_1 + m_2} = \frac{m_2 v_1 - m_2 v_2}{m_1 + m_2}$$

$$\implies \vec{v}_1' = \frac{m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2) \quad \textit{and} \quad \vec{v}_2' = \frac{m_1}{m_1 + m_2}(\vec{v}_2 - \vec{v}_1). \qquad (6)$$

(We get the second equation by switching the labels 1 and 2, since there isn't anything special about the labels.)

Kinetic Energy

Using (6), the initial kinetic energy in the COM frame can be written

$$\begin{aligned} E'_{Ki} &= \frac{1}{2}m_1 u'^2_1 + \frac{1}{2}m_2 u'^2_2 \\ &= \frac{1}{2}m_1 \left| \frac{m_2}{m_1+m_2}(\vec{u}_1 - \vec{u}_2) \right|^2 + \frac{1}{2}m_2 \left| \frac{m_1}{m_1+m_2}(\vec{u}_2 - \vec{u}_1) \right|^2 \\ &= \frac{1}{2}\frac{m_1 m_2^2}{m_1+m_2}\left|\vec{u}_1 - \vec{u}_2\right|^2 + \frac{1}{2}\frac{m_2 m_1^2}{m_1+m_2}\left|\vec{u}_2 - \vec{u}_1\right|^2 \end{aligned}$$

Because $|\vec{u}_1 - \vec{u}_2| = |\vec{u}_2 - \vec{u}_1|$, we can factor it out:

$$\begin{aligned} E'_{Ki} &= \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)^2}|\vec{u}_1 - \vec{u}_2|^2(m_2 + m_1) \\ \implies E'_{Ki} &= \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}|\vec{u}_1 - \vec{u}_2|^2 \end{aligned}$$

And similarly,

$$E'_{Kf} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}|\vec{v}_1 - \vec{v}_2|^2.$$

The fraction of kinetic energy that remains after a collision can be written in a simplified form,

$$\frac{E'_{Kf}}{E'_{Ki}} = \left(\frac{|\vec{v}_1 - \vec{v}_2|}{|\vec{u}_1 - \vec{u}_2|}\right)^2 = e^2, \quad (7)$$

where $e = \dfrac{|\vec{v}_1 - \vec{v}_2|}{|\vec{u}_1 - \vec{u}_2|}$ is called the *coefficient of restitution.*

If $e = 1$, then $E'_{Kf} = E'_{Ki}$ and we have an *elastic collision*. The smallest this can get is when $\vec{v}_1 = \vec{v}_2$ and the two objects move together after the collision and the ratio is 0: this is a *maximally inelastic collision.* If $0 < e < 1$, then we have a *partially inelastic collision*, while if $e > 1$ we have a *superelastic collision:* the total energy increases during the collision, like in an explosion. Specifying the value of e gives us the second equation we need to solve for both final velocities in a collision.

NOTE: I gave the wrong definition of e in class.

Elastic Collision

For example, suppose no energy is lost, and $e = 1$. That means that

$$|\vec{v}_1 - \vec{v}_2| = |\vec{u}_1 - \vec{u}_2|.$$

Now these are the velocities in the lab frame. However, if we remember that $\vec{v}\,'_1 = \vec{v}_1 - \vec{v}_{com} \implies \vec{v}_1 = \vec{v}\,'_1 + \vec{v}_{com}$ and similarly for all velocities, we can write

$$|\vec{v}_1 - \vec{v}_2| = |(\vec{v}\,'_1 + \vec{v}_{com}) - (\vec{v}\,'_2 - \vec{v}_{com})| = |\vec{v}\,'_1 - \vec{v}\,'_2|$$

so we can use the velocities in the COM frame as well. Now, in the COM frame, we know from Eq. (5) that $\vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1$, and so

$$|\vec{v}\,'_1 - \vec{v}\,'_2| = |\vec{v}\,'_1 - -\frac{m_1}{m_2}\vec{v}\,'_1| = \left(1 + \frac{m_1}{m_2}\right)|\vec{v}\,'_1|$$

and similarly for $\vec{u}\,'$. Since they must be equal, we end up with

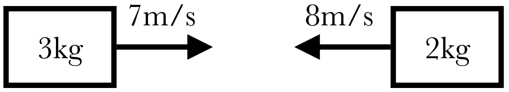
$$\left(1 + \frac{m_1}{m_2}\right)|\vec{v}\,'_1| = \left(1 + \frac{m_1}{m_2}\right)|\vec{u}\,'_1|$$

$$\implies |\vec{v}\,'_1| = |\vec{u}\,'_1|.$$

In other words, in the COM frame, the speed of the first block is the same after the collision as it is before the collision; only the direction changes, and the same is true for the second block. In a one-dimensional collision, that means that both blocks maintain the same speed (in the COM frame), but in the opposite directions.

This suggests a method for solving an elastic collision problem.

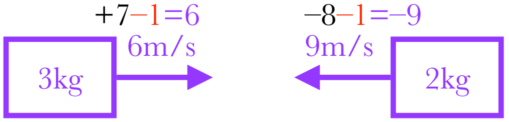
1. Find $\vec{v}_{com} = \vec{p}_{tot}/m_{tot}$.



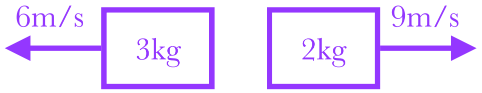
$$\vec{p}_1 = +21 \qquad \vec{p}_2 = -16$$

$$\vec{v}_{com} = \frac{21 - 16}{5} = +1$$

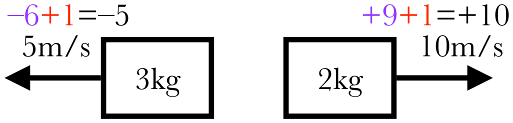
2. Subtract $\vec{v}_{com}$ from both velocities to switch into the COM frame. (Remember that vectors to the left are already negative.)



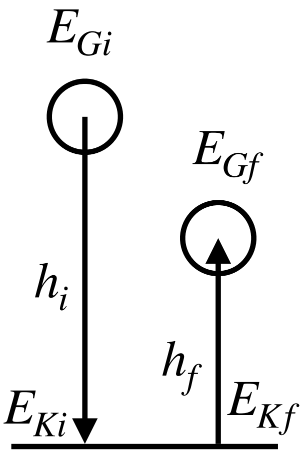
3. Flip the COM velocities.



4. Add $\vec{v}_{com}$ back to switch back into lab frame.



Bouncy Ball



One example where the coefficient of restitution is useful is when you bounce a ball on the ground and watch how high it rebounds. Suppose the ball is dropped from an initial height h_i, starting with gravitational energy $E_{Gi} = mgh_i$. As it falls, this energy turns into kinetic energy, and right before it hits the floor it has kinetic energy $E_{Ki} = mgh_i$. Right after it bounces, it has less kinetic energy E_{Kf}, and as the ball rises it eventually all becomes gravitational energy, so at its highest point $E_{Gf} = E_{Kf}$ and so $E_{Kf} = mgh_f$.

Now using Equation 7, we see that

$$e^2 = \frac{E_{Kf}}{E_{Ki}} = \frac{mgh_f}{mgh_i} = \frac{h_f}{h_i},$$

and so after one bounce the height of the ball drops to the level $e^2 h_i$, and if the ball keeps bouncing, its height keeps being reduced by the same amount, so $e^4 h_i$, $e^6 h_i$, etc. If we want to find the coefficient of restitution, we can take the square root of both sides to find

$$e = \sqrt{\frac{h_f}{h_i}}.$$

Summary

So what do you actually need to know to succeed with homework problems and exams?

- Know what the center of mass is and how to calculate it (Eq 3)
- Know that $p_{tot} = m_{tot} v_{com}$ and what v_{com} is (Eq 4)
- Know what the coefficient of restitution *e* is, and the different types of collisions for different values of *e*.
- Know that $e = \dfrac{|v_f - v_i|}{|u_f - u_i|}$ and $e^2 = \dfrac{E_{Kf}}{E_{Ki}}$ (Eq 7)
- Be able to solve an elastic collision problem by shifting from the lab frame to the center-of-mass frame and back.
- Know how to solve a maximally inelastic problem. (This doesn't involve a COM shift.)
- You need the bouncy ball result for a homework problem at least.

You do not need to be able to do the derivations or explain where the equations come from.